Limits of betweenness relations

Logic Colloquium 2016, Leeds

David Bradley-Williams

University of Central Lancashire, Preston

August 2016

Jordan Groups

Definition

Let (G, Ω) be a transitive permutation group. A **Jordan set** is a subset $\Gamma \subset \Omega$ with $|\Gamma| > 1$ such that $(G_{(\Omega \setminus \Gamma)}, \Gamma)$ is transitive.

Example (Improper Jordan sets)

- Ω is a Jordan set.
- If G is $n+1$-transitive and $\Theta \subset \Omega$ a finite set of n points, then $\Omega \setminus \Theta$ is a Jordan set.

Definition

A Jordan set is **proper** if it is non-trivial and neither of the above.

Definition

If G admits a proper Jordan set then we call it a **Jordan group**.

Definition

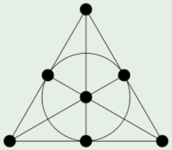
Let (G, Ω) be a transitive permutation group. A **Jordan set** is a subset $\Gamma \subset \Omega$ with $|\Gamma| > 1$ such that $(G_{(\Omega \setminus \Gamma)}, \Gamma)$ is transitive.

Definition

If G admits a proper Jordan set then we call it a **Jordan group**.

Example

- $\mathrm{Aut}(\mathbb{Q}, <)$ is a primitive Jordan group.
  Open intervals $(a, b) \subseteq \mathbb{Q}$ are proper Jordan sets.
- $PGL(3, 2) = \mathrm{Aut}(PG(2, 2),)$ is a finite, primitive, 2-transitive Jordan group. Jordan sets are complements of projective lines.



Jordan sets

Fact (Basic properties of Jordan sets)

Let Γ_1 and Γ_2 be proper Jordan sets for (G, Ω). Then

- *All translates of Γ_1 are proper Jordan sets;*
- *If $\Gamma_1 \cap \Gamma_2 \neq \emptyset$ then $\Gamma_1 \cup \Gamma_2$ is a Jordan set.*

Jordan sets

Fact (Basic properties of Jordan sets)

Let Γ_1 *and* Γ_2 *be proper Jordan sets for* (G, Ω)*. Then*

- *All translates of* Γ_1 *are proper Jordan sets;*
- *If* $\Gamma_1 \cap \Gamma_2 \neq \emptyset$ *then* $\Gamma_1 \cup \Gamma_2$ *is a Jordan set.*

A Jordan group has a family of Jordan sets.

Jordan sets

Fact (Basic properties of Jordan sets)

Let Γ_1 and Γ_2 be proper Jordan sets for (G, Ω). Then

- *All translates of Γ_1 are proper Jordan sets;*
- *If $\Gamma_1 \cap \Gamma_2 \neq \emptyset$ then $\Gamma_1 \cup \Gamma_2$ is a Jordan set.*

A Jordan group has a family of Jordan sets.

Definition

(G, Ω) is **primitive** if it preserves no non-trivial, proper equivalence relation on Ω.

Jordan sets

Fact (Basic properties of Jordan sets)

Let Γ_1 and Γ_2 be proper Jordan sets for (G, Ω). Then

- *All translates of Γ_1 are proper Jordan sets;*
- *If $\Gamma_1 \cap \Gamma_2 \neq \emptyset$ then $\Gamma_1 \cup \Gamma_2$ is a Jordan set.*

A Jordan group has a family of Jordan sets.

Definition

(G, Ω) is **primitive** if it preserves no non-trivial, proper equivalence relation on Ω.

Definition

A Jordan set Γ is **primitive** if $(G_{(\Omega \setminus \Gamma)}, \Gamma)$ is primitive (not just transitive).

Semilinear ordering

Definition

A partial ordering $(\Omega, <)$ is **semilinear** if

- Every two points have some lower bound;
- For every a, $L_a = \{x \leq a\}$ is linear;
- $(\Omega, <)$ is not linear.

Semilinear ordering

Definition

A partial ordering $(\Omega, <)$ is **semilinear** if

- Every two points have some lower bound;
- For every a, $L_a = \{x \leq a\}$ is linear;
- $(\Omega, <)$ is not linear.

Example (on board)

Construction of $(Q; 2, +)$ yields a countable semilinear ordering $(\Omega, <)$.

Semilinear ordering

Definition

A partial ordering $(\Omega, <)$ is **semilinear** if

- Every two points have some lower bound;
- For every a, $L_a = \{x \leq a\}$ is linear;
- $(\Omega, <)$ is not linear.

Example (on board)

Construction of $(Q; 2, +)$ yields a countable semilinear ordering $(\Omega, <)$.

Definition (via Adeleke-Neumann)

A **(general) betweenness relation** is a structure $(\Omega, B(y; x, z))$ derived from a semilinear ordering.

$$B(y; x, z) :\Leftrightarrow y \text{ is on the interval } [x, z].$$

Invariant structures for Jordan groups

Theorem (Adeleke, Macpherson (1996))

Let (G, Ω) *be an infinite, primitive Jordan group which is not highly-transitive on* Ω*. Then* G *preserves on* Ω *one of the following kinds of structure:*

Invariant structures for Jordan groups

Theorem (Adeleke, Macpherson (1996))

Let (G, Ω) be an infinite, primitive Jordan group which is not highly-transitive on Ω. Then G preserves on Ω one of the following kinds of structure:

1. *Dense linear–like (DLO or first order reduct);*

Invariant structures for Jordan groups

Theorem (Adeleke, Macpherson (1996))

Let (G, Ω) be an infinite, primitive Jordan group which is not highly-transitive on Ω. Then G preserves on Ω one of the following kinds of structure:

1. *Dense linear–like (DLO or first order reduct);*
2. *Dense 'tree–like' (structures related to dense semilinear orderings);*

Invariant structures for Jordan groups

Theorem (Adeleke, Macpherson (1996))

Let (G, Ω) be an infinite, primitive Jordan group which is not highly-transitive on Ω. Then G preserves on Ω one of the following kinds of structure:

1. *Dense linear–like (DLO or first order reduct);*
2. *Dense 'tree–like' (structures related to dense semilinear orderings);*
3. *A Steiner system (combinatorial geometries);*

Invariant structures for Jordan groups

Theorem (Adeleke, Macpherson (1996))

Let (G, Ω) be an infinite, primitive Jordan group which is not highly-transitive on Ω. Then G preserves on Ω one of the following kinds of structure:

1. *Dense linear–like (DLO or first order reduct);*
2. *Dense 'tree–like' (structures related to dense semilinear orderings);*
3. *A Steiner system (combinatorial geometries);*
4. *A 'limit' of Steiner systems;*

Invariant structures for Jordan groups

Theorem (Adeleke, Macpherson (1996))

Let (G, Ω) be an infinite, primitive Jordan group which is not highly-transitive on Ω. Then G preserves on Ω one of the following kinds of structure:

1. *Dense linear–like (DLO or first order reduct);*
2. *Dense 'tree–like' (structures related to dense semilinear orderings);*
3. *A Steiner system (combinatorial geometries);*
4. *A 'limit' of Steiner systems;*
5. *'Limits' of certain tree–like structures.*

Invariant structures for Jordan groups

Theorem (Adeleke, Macpherson (1996))

Let (G, Ω) be an infinite, primitive Jordan group which is not highly-transitive on Ω. Then G preserves on Ω one of the following kinds of structure:

1. *Dense linear–like (DLO or first order reduct);*
2. *Dense 'tree–like' (structures related to dense semilinear orderings);*
3. *A Steiner system (combinatorial geometries);*
4. *A 'limit' of Steiner systems;*
5. *'Limits' of certain tree–like structures.*

Moreover, if G has a primitive Jordan set, then it preserves a more familiar structure: of type (1) or (2) above (Adeleke, Neumann 1996).

Invariant structures for Jordan groups

Theorem (Adeleke, Macpherson (1996))

Let (G, Ω) be an infinite, primitive Jordan group which is not highly-transitive on Ω. Then G preserves on Ω one of the following kinds of structure:

1. *Dense linear–like (DLO or first order reduct);*
2. *Dense 'tree–like' (structures related to dense semilinear orderings);*
3. *A Steiner system (combinatorial geometries);*
4. *A 'limit' of Steiner systems;*
5. *'Limits' of certain tree–like structures.*

Moreover, if G has a primitive Jordan set, then it preserves a more familiar structure: of type (1) or (2) above (Adeleke, Neumann 1996).

Note: Their list is more precise.

Applications

Classifications of **Finite** Jordan groups turned up in the the study of $\aleph_0$-categorical, $\aleph_0$-stable structures by Cherlin, Harrington and Lachlan (1985) (simultaneously Zilber).

Applications

Classifications of **Finite** Jordan groups turned up in the the study of $\aleph_0$-categorical, $\aleph_0$-stable structures by Cherlin, Harrington and Lachlan (1985) (simultaneously Zilber).

Descriptions of **infinite** Jordan groups have found applications.

- Describing cycle types in infinite permutation groups, Macpherson-Praeger (1995);
- Permutation groups with infinite bounded supports, Adeleke-Neumann(1996);

Applications

Classifications of **Finite** Jordan groups turned up in the the study of $\aleph_0$-categorical, $\aleph_0$-stable structures by Cherlin, Harrington and Lachlan (1985) (simultaneously Zilber).

Descriptions of **infinite** Jordan groups have found applications.

- Describing cycle types in infinite permutation groups, Macpherson-Praeger (1995);
- Permutation groups with infinite bounded supports, Adeleke-Neumann(1996);
- Classifications of topologically closed groups of Sym($\mathbb{N}$) containing certain automorphism groups: Bodirsky-Macpherson, Kaplan-Simon, B-W.

Limits of B-relations

The definition is discovered by Adeleke and Macpherson (1996), technical conditions on the permutation group to be approximated by quotients.

Limits of B-relations

The definition is discovered by Adeleke and Macpherson (1996), technical conditions on the permutation group to be approximated by quotients.

Idea of a limit of B-relations

Such a group (G, Ω) is an infinite, primitive Jordan group with no primitive Jordan sets. There is:

- An infinite chain of subsets $\Gamma_i \subset \Gamma_j \subset \Omega$;
- $\bigcup \Gamma_i = \Omega$;

Limits of B-relations

The definition is discovered by Adeleke and Macpherson (1996), technical conditions on the permutation group to be approximated by quotients.

Idea of a limit of B-relations

Such a group (G, Ω) is an infinite, primitive Jordan group with no primitive Jordan sets. There is:

- An infinite chain of subsets $\Gamma_i \subset \Gamma_j \subset \Omega$;
- $\bigcup \Gamma_i = \Omega$;
- Corresponding chain of subgroups $H_i = G_{(\Omega \setminus \Gamma_i)}$;

Limits of B-relations

The definition is discovered by Adeleke and Macpherson (1996), technical conditions on the permutation group to be approximated by quotients.

Idea of a limit of B-relations

Such a group (G, Ω) is an infinite, primitive Jordan group with no primitive Jordan sets. There is:

- An infinite chain of subsets $\Gamma_i \subset \Gamma_j \subset \Omega$;
- $\bigcup \Gamma_i = \Omega$;
- Corresponding chain of subgroups $H_i = G_{(\Omega \setminus \Gamma_i)}$;
- (H_i, Γ_i) are **imprimitive** with maximal congruence ρ_i;

Limits of B-relations

The definition is discovered by Adeleke and Macpherson (1996), technical conditions on the permutation group to be approximated by quotients.

Idea of a limit of B-relations

Such a group (G, Ω) is an infinite, primitive Jordan group with no primitive Jordan sets. There is:

- An infinite chain of subsets $\Gamma_i \subset \Gamma_j \subset \Omega$;
- $\bigcup \Gamma_i = \Omega$;
- Corresponding chain of subgroups $H_i = G_{(\Omega \setminus \Gamma_i)}$;
- (H_i, Γ_i) are **imprimitive** with maximal congruence ρ_i;
- H_i preserves a **betweenness relation** on the quotient Γ_i / ρ_i;

Limits of B-relations

The definition is discovered by Adeleke and Macpherson (1996), technical conditions on the permutation group to be approximated by quotients.

Idea of a limit of B-relations

Such a group (G, Ω) is an infinite, primitive Jordan group with no primitive Jordan sets. There is:

- An infinite chain of subsets $\Gamma_i \subset \Gamma_j \subset \Omega$;
- $\bigcup \Gamma_i = \Omega$;
- Corresponding chain of subgroups $H_i = G_{(\Omega \setminus \Gamma_i)}$;
- (H_i, Γ_i) are **imprimitive** with maximal congruence ρ_i;
- H_i preserves a **betweenness relation** on the quotient Γ_i / ρ_i;
- These (H_i, Γ_i) approximate (G, Ω),

Limits of B-relations

The definition is discovered by Adeleke and Macpherson (1996), technical conditions on the permutation group to be approximated by quotients.

Idea of a limit of B-relations

Such a group (G, Ω) is an infinite, primitive Jordan group with no primitive Jordan sets. There is:

- An infinite chain of subsets $\Gamma_i \subset \Gamma_j \subset \Omega$;
- $\bigcup \Gamma_i = \Omega$;
- Corresponding chain of subgroups $H_i = G_{(\Omega \setminus \Gamma_i)}$;
- (H_i, Γ_i) are **imprimitive** with maximal congruence ρ_i;
- H_i preserves a **betweenness relation** on the quotient Γ_i / ρ_i;
- These (H_i, Γ_i) approximate (G, Ω),
- e.g. their maximal congruences ρ_i refine as i increases.

Constructions of these limits

Theorem (Adeleke (90's/2013))

There is an infinite group (G, Ω) *which preserves a limit of B-relations.*

Constructions of these limits

Theorem (Adeleke (90's/2013))

There is an infinite group (G, Ω) which preserves a limit of B-relations.

This is by construction, taking a direct limit of infinite groups. The group is not 3-homogeneous, so any preserved structure is not $\aleph_0$-categorical.

Constructions of these limits

Theorem (Adeleke (90's/2013))

There is an infinite group (G, Ω) which preserves a limit of B-relations.

This is by construction, taking a direct limit of infinite groups. The group is not 3-homogeneous, so any preserved structure is not $\aleph_0$-categorical.

Theorem (Bhattacharjee-Macpherson (2006))

There is an $\aleph_0$-categorical structure $M = (\Omega, L(x; y, z))$ whose automorphism group preserves a limit of B-relations.

Constructions of these limits

Theorem (Adeleke (90's/2013))

There is an infinite group (G, Ω) which preserves a limit of B-relations.

This is by construction, taking a direct limit of infinite groups. The group is not 3-homogeneous, so any preserved structure is not $\aleph_0$-categorical.

Theorem (Bhattacharjee-Macpherson (2006))

There is an $\aleph_0$-categorical structure $M = (\Omega, L(x; y, z))$ whose automorphism group preserves a limit of B-relations.

The construction of M involves dual notions of a **tree of B-sets** and a class $\mathscr{C}$ of finite first order structures.

Constructions of these limits

Theorem (Adeleke (90's/2013))

There is an infinite group (G, Ω) which preserves a limit of B-relations.

This is by construction, taking a direct limit of infinite groups. The group is not 3-homogeneous, so any preserved structure is not $\aleph_0$-categorical.

Theorem (Bhattacharjee-Macpherson (2006))

There is an $\aleph_0$-categorical structure $M = (\Omega, L(x; y, z))$ whose automorphism group preserves a limit of B-relations.

The construction of M involves dual notions of a **tree of B-sets** and a class $\mathscr{C}$ of finite first order structures.

Generalised Fraïssé amalgamation in $\mathscr{C}$ results in structure M^* with a natural reduct to the desired M.

Generalised Fraïssé Limits

Definition

General amalgamation classes $(\mathscr{K}, \mathscr{E})$

- $\mathscr{K}$ is a countable class of finite structures containing $\emptyset$;
- $\mathscr{E}$ is a countable collection of **strong** embeddings between structures in $\mathscr{K}$;
- $(\mathscr{K}, \mathscr{E})$ has **AP($\mathscr{E}$)**, amalgamation property for strong embeddings;
- For all A in $\mathscr{K}$,
    - $\emptyset$ is strongly embedded in A
    - A is strongly embedded in A.

Generalised Fraïssé Limits

Definition

General amalgamation classes $(\mathscr{K}, \mathscr{E})$

- $\mathscr{K}$ is a countable class of finite structures containing $\emptyset$;
- $\mathscr{E}$ is a countable collection of **strong** embeddings between structures in $\mathscr{K}$;
- $(\mathscr{K}, \mathscr{E})$ has **AP($\mathscr{E}$)**, amalgamation property for strong embeddings;
- For all A in $\mathscr{K}$,
  - $\emptyset$ is strongly embedded in A
  - A is strongly embedded in A.

Definition

A Fraïssé limit of a general amalgamation class $(\mathscr{K}, \mathscr{E})$ is the union $\bigcup(A_i)$ of a **rich** sequence

$$A_1 \rightarrow_{e_1} A_2 \rightarrow_{e_2} \ldots \rightarrow_{e_{i-1}} A_i \rightarrow_{e_i} \ldots$$

Theorem (Fraïssé, Jónsson, Shelah, Hrushovski, Evans, Ziegler,...)

Suppose $(\mathscr{K}, \mathscr{E})$ *is a general amalgamation class. Then rich sequences exist and the Fraïssé limit* M *is unique up to isomorphism.*

Theorem (Fraïssé, Jónsson, Shelah, Hrushovski, Evans, Ziegler,...)

Suppose $(\mathscr{K}, \mathscr{E})$ is a general amalgamation class. Then rich sequences exist and the Fraïssé limit M is unique up to isomorphism.
Moreover, if A and B are strongly embedded in M and $p : A \to B$ an isomorphism, then p extends to an automorphism of M.

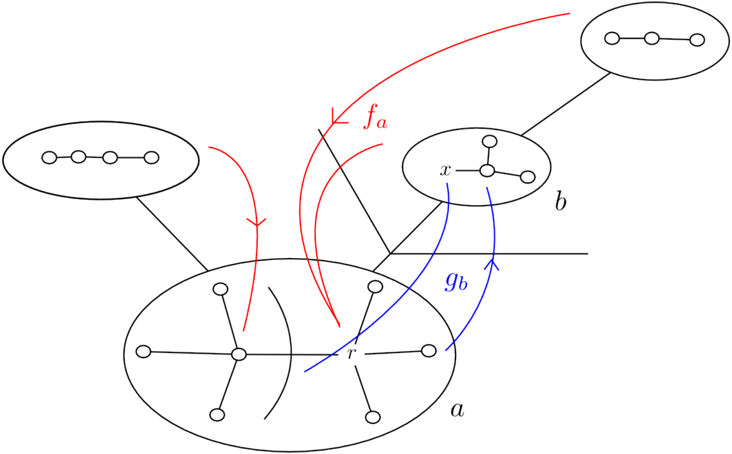
Theorem (Fraïssé, Jónsson, Shelah, Hrushovski, Evans, **Ziegler**,...)

Suppose $(\mathscr{K}, \mathscr{E})$ *is a general amalgamation class. Then rich sequences exist and the Fraïssé limit* M *is unique up to isomorphism.*
Moreover, if A *and* B *are strongly embedded in* M *and* $p : A \to B$ *an isomorphism, then* p *extends to an automorphism of* M*.*

Why?

One reason is that such classes need not have HP.
The structure induced on a subset might not be in the class you wish to amalgamate.

Trees of B-sets



An illustrated example of a tree of B-sets.

Trees of B-sets

Fix $(T, <)$ to be a particular semilinear order: the $\mathbb{N}$–tree (see board).

Trees of B-sets

Fix $(T, <)$ to be a particular semilinear order: the $\mathbb{N}$–tree (see board).

Definition

A **finite tree of B-sets**, A, consists of

- A finite, convex, meet-closed subtree $T^A \leq T$;

Trees of B-sets

Fix $(T, <)$ to be a particular semilinear order: the $\mathbb{N}$–tree (see board).

Definition

A **finite tree of B-sets**, A, consists of

- A finite, convex, meet-closed subtree $T^A \leq T$;
- For each $t \in T^A$ a B-set $(B(t), B_t)$ subject to the following.
  - For each $t \in T^A$ such that $|B(t)| > 1$, there is a surjection

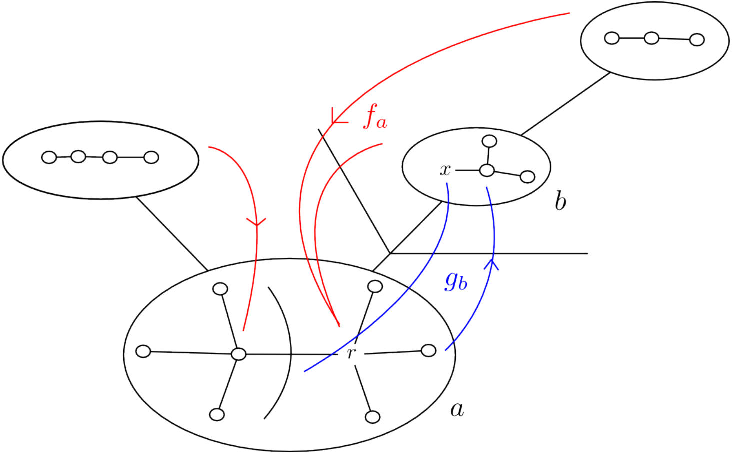
    $$f_t : \{s \in T^A : s > t\} \to B(t),$$

    such that f_t^{-1} is a bijection $B(t) \to \{\text{cones above } t \text{ in } T^A\}$;
  - For every $s \in \mathrm{succ}_{A^T}(t)$ a surjection

    $$g_{ts} : B(t) \setminus \{f_t(s)\} \to B(s),$$

    such that g_{ts}^{-1} is a bijection $\{\text{branches at } f_t(s)\} \to B(s)$.

Trees of B-sets



An illustrated example of a tree of B-sets.

Arboreal Isomorphisms

Definition

An **arboreal isomorphism** between finite trees of B-sets A, A' is

$$(\tau, \{\phi_s\}) : A \to A',$$

where:

- $\tau : T \to T$ is an automorphism (of all) of $(T, <)$ taking T_A to $T_{A'}$;
- For each $s \in T^A$ a B-set isomorphism

  $$\phi_s : B(s) \to B(\tau(s))$$

  that commutes with all the f_t and g_{st}.

Theorem (B-W)

There is a way to regard the class of finite B-sets over T as a class $(\mathscr{K}, \mathscr{E})$ where $\mathscr{E}$ are the embeddings induced by arboreal ismorphisms between finite B-sets over T. Then $(\mathscr{K}, \mathscr{E})$ is a general amalgamation class.

Thank you!