Reducts of Primitive Jordan Structures

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Reducts

Definition

Let $\mathscr{M}$ be a first-order structure with universe Ω. A relational structure $\mathscr{N}$ is a **reduct** of $\mathscr{M}$ if

- $\mathscr{N}$ has universe Ω;
- Every relation of $\mathscr{N}$ is $\emptyset$-definable in $\mathscr{M}$.

Definition

- If $\mathscr{M}$ is also a reduct of $\mathscr{N}$ then $\mathscr{M}$ and $\mathscr{N}$ are **interdefinable**.
- Otherwise $\mathscr{N}$ is a **proper** reduct of $\mathscr{M}$.

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Theorem (Cameron)

Let $\mathscr{M} = (\mathbb{Q}, <)$ and $\mathscr{N}$ a reduct of $\mathscr{M}$. Then $\mathscr{N}$ is interdefinable with

- $\mathscr{M} = (\mathbb{Q}, <)$;
- $(\mathbb{Q}, bet)$, *where* $bet(x, y, z)$ *is betweenness on* $\mathbb{Q}$;
- $(\mathbb{Q}, circ)$, *where* $circ(x, y, z)$ *is the circular ordering of* $\mathbb{Q}$;
- $(\mathbb{Q}, sep)$, *where* $sep(x, y, z, w)$ *is the separation relation on* $\mathbb{Q}$;
- $(\mathbb{Q}, =)$, *the trivial structure.*

Topology on Sym(Ω)

Fix Ω a countably infinite set.

Definition

The **pointwise convergence** topology on Sym(Ω) is inherited from Ω^Ω.

This is generated by a basis of clopen sets

$$[g]_A = \{h : h_{\restriction A} = g_{\restriction A}\},$$

for finite $A \subseteq$ Sym(Ω).

Fact

- *Closed subgroups of* Sym(Ω) *are automorphism groups of relational structures.*

ω-categorical structures

Definition

$\mathscr{M}$ is ω-**categorical** if every countable $\mathscr{N} \models \mathrm{Th}(\mathscr{M})$ is isomorphic to $\mathscr{M}$.

Theorem (Ryll-Nardzewski)

$\mathscr{M}$ is ω-categorical if and only if $(\mathrm{Aut}(M), \Omega)$ is **oligomorphic**.

Fact (M is ω-categorical)

Let G be a closed subgroup of $\mathrm{Sym}(\Omega)$,
$\mathscr{N}$ a relational structure with $G = \mathrm{Aut}(\mathscr{N})$.
Then $\mathscr{N}$ is a (proper) reduct of $\mathscr{M}$ iff G (properly) contains $\mathrm{Aut}(\mathscr{M})$.

Example

- Cameron - 5 reducts of $(\mathbb{Q}, <)$;
- Thomas - 5 reducts of the Random Graph;
- Thomas - No proper reducts of the generic K_n-free graphs;
- Bennett - 5 reducts of the generic tournament;
- Junker-Ziegler - 116 reducts of $(\mathbb{Q}, <, 0)$;
- Pach-Pinsker-Pluhár-Pongrácz-Szabó - 5 reducts of generic partial order;
- Bodirsky-Pinsker-Pongrácz - 42 proper reducts of the generic ordered graph;
- Agarwal - 11 reducts of generic digraph;
- Agarwal-Kompatscher - uncountably many Henson digraphs with no proper reducts.

Definition

M is **homogeneous** if every isomorphism between finite substructures extends to an automorphism.

Definition

M is **finitely homogeneous** if it is homogeneous in some finite relational language.

Conjecture (Thomas)

If M is finitely homogeneous then it has only finitely many reducts up to interdefinability.

We seek some general progress towards this!

Semilinear orderings

Definition

A partial ordering $(\Omega, <)$ is **semilinear** if

- Every two points have some lower bound;
- For every a, $L_a = \{x \leq a\}$ is linear;
- $(\Omega, <)$ is not linear.

Example

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$\mathscr{M}$ is **relatively 2-transitive** if every isomorphism between 2-element sub-structures extends to an automorphism.

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Example

$(Q, 2, +)$ is a relatively 2-transitive semilinear ordering, not homogeneous.

Theorem (Droste)

Let $\mathscr{M} = (\Omega, <)$ be a countable, relatively 2-transitive semilinear ordering. Then $\mathscr{M}$ is isomorphic to $(\mathbb{Q}, k, +/-)$ for some choice of branching order $k \in \mathbb{N} \cup \{\aleph_0\}$ and choice of $+$ or $-$. They are pairwise non-isomorphic.

Droste calls these **2-homogeneous trees**.

Theorem (Droste-Holland-Macpherson)

Every such $(\mathbb{Q}, k, +/-)$ is finitely homogeneous (in a ternary language). Hence they are all ω-categorical.

Definition

Let $\mathscr{M} = (\Omega, <)$ be a semilinear ordering. For $x, z \in \Omega$ define the **path** from x to z

$$[x, z] := L_x \triangle L_z (\cup\{x \wedge z\}).$$

The **betweenness relation** in $\mathscr{M}$ is given by:

$$B(x, y, z) \Leftrightarrow y \in [x, z].$$

$\mathrm{Aut}(\Omega, B)$ is the collection of **re-rootings** of $\mathscr{M}$.

Theorem (Bodirsky-BW-Pinsker-Pongrácz)

Let $\mathscr{M} = (\Omega, <)$ *be the semilinear ordering* $(\mathbb{Q}, 2, -)$ *and* $\mathscr{N}$ *a reduct. Then* $\mathscr{N}$ *is interdefinable with*

- $\mathscr{M} = (\Omega, <)$;
- (Ω, B);
- $(\Omega, =)$.

A corollary to classifying the **model complete cores** of reducts of $\mathscr{M}$.

Theorem (BW)

Let $\mathscr{M} = (\Omega, <)$ *be a relatively 2-transitive semilinear ordering, that is any* $(\mathbb{Q}, k, +/-)$, *and* $\mathscr{N}$ *a reduct.*
Then $\mathscr{N}$ *is interdefinable with*

- $\mathscr{M} = (\Omega, <)$;
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Jordan Groups

Definition

Let (G, Ω) be a transitive permutation group. A **Jordan set** is a subset $\Gamma \subset \Omega$ with $|\Gamma| > 1$ such that $(G_{(\Omega \setminus \Gamma)}, \Gamma)$ is transitive.

Example

- Ω is a Jordan set.
- If G is $n + 1$-transitive and $\Theta \subset \Omega$ a finite set of n points, then $\Omega \setminus \Theta$ is a Jordan set.

Definition

A Jordan set is **proper** if it is non-trivial and neither of the above.

Definition

If G admits a proper Jordan set then we call it a **Jordan group**.

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Definition

Call $\mathcal{N}$ a **Jordan** structure if $(\mathrm{Aut}(\mathcal{N}), \Omega)$ is a Jordan group.

Example

- $\mathrm{Aut}(\mathbb{Q}, <)$ is a primitive Jordan group.
  Jordan sets are open intervals.
- A relatively 2-transitive semilinear order is a primitive Jordan structure. Cones are primitive Jordan sets (Adeleke-Neumann).

Jordan sets

Fact (Basic properties of Jordan sets)

Let Γ_1 and Γ_2 be proper Jordan sets for (G, Ω). Then

- *All translates of Γ_1 are proper Jordan sets;*
- *If $\Gamma_1 \cap \Gamma_2 \neq \emptyset$ then $\Gamma_1 \cup \Gamma_2$ is a proper Jordan set.*

We have a family of Jordan sets.

Fact

Let $G \leq H \leq \mathrm{Sym}(\Omega)$, and Γ a proper (primitive) Jordan set for G. Then Γ is a (primitive) Jordan set for H.

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Fact

Reducts of primitive Jordan structures are primitive Jordan structures!

Primitive Jordan groups

Theorem (Adeleke-Neumann)

Let (G, Ω) be an infinite, primitive Jordan group which is not highly-transitive on Ω with a primitive Jordan set. Then G preserves on Ω one of the following kinds of structure:

1. *Dense linear ordering (or first order reduct);*
2. *Dense semilinear ordering;*
3. *Dense B-relation;*
4. *Proper C-relation;*
5. *Proper D-relation.*

These relational structures are studied and axiomatised by Adeleke-Neumann.

Other results using Jordan groups

Theorem (Bodirsky-Macpherson)

Bodirsky and Macpherson constructed a D-relation such that (Ω, D) *is a primitive Jordan structure, is not* ω*-categorical and has* **no proper, non-trivial reducts** *and* $\mathrm{Aut}(\Omega, D)$ *is* **maximally closed** *in* $\mathrm{Sym}(\Omega)$.

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Theorem (Kaplan-Simon)

Let V be a vector space of dimension $n \geq 3$ over the field $\mathbb{Q}$. So $AGL(n, \mathbb{Q}) = \mathrm{Aut}(V, f)$ with $f(x, y, z) = x + y - z$. Then

- *(V, f) has no proper, non-trivial reducts;*
- *$AGL(n, \mathbb{Q})$ is maximally closed in $\mathrm{Sym}(\Omega)$.*

Similar result for $\mathrm{PGL}(n, \mathbb{Q})$.

Work to do

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Classify primitive, countable Jordan semilinear orders.

- Consider weakly 2-transitive trees of Droste-Holland-Macpherson.
- Consider the classification of countably 1-transitive trees by Truss-Chicot.

Conjecture

Every countable, finitely homogeneous, binary-language, primitive Jordan structure has only finitely many reducts.