Canonical Invariants for t-stratifications,

or canonical invariants for definable sets in $\mathbb{K}((t))$

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Valuations on Laurent Series over $\mathbb{R}$ and $\mathbb{C}$

- Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$.
- The **field of (formal) Laurent series** over $\mathbb{K}$ is

$$\mathbb{K}((t)) = \left\{ \sum_{N \leq r \in \mathbb{Z}} a_r t^r \mid a_r \in \mathbb{K}, a_N \neq 0 \right\}.$$

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Definition

The *t*-**adic valuation** $|.|$ on $\mathbb{K}((t))$ is defined

- $|0| = 0$
- $|\sum_{N \leq r \in \mathbb{Z}} a_r t^r| = \gamma^N$, where $a_N \neq 0$ and $\gamma \in (0, 1)$.

$\mathscr{O}, \mathscr{M}$

Definition

The t-**adic valuation** $|.|$ on $\mathbb{K}((t))$:

- $|0| = 0$,
- $|\sum_{r \geq N \in \mathbb{Z}} a_r t^r| = \gamma^N$, where $a_N \neq 0$ and $\gamma \in (0,1)$.

- Valuation ring: $\mathscr{O} = \{\sum_{0 \leq r \in \mathbb{Z}} a_r t^r\} = \{x : |x| \leq 1\}$,
- Maximal ideal: $\mathscr{M} = t.\mathscr{O} = \{x : |x| < 1\}$,
- Residue field: $\operatorname{res}(\mathbb{K}((t))) = \mathscr{O}/\mathscr{M} \cong \mathbb{K}$,
- Value group: $\Gamma = |\mathbb{K}((t))|$.

$\mathbb{K}((t))^n$

- Valued field in 3-sorted language: $(\mathbb{K}((t)), \mathbb{K}, \Gamma)$.
- We are interested in definable subsets of $\mathbb{K}((t))^n$.
- For $v = (v_1, ..., v_n) \in \mathbb{K}((t))^n$ set

$$\|v\| = \max\{|v_i|\}.$$

- Makes $\mathbb{K}((t))^n$ an **ultrametric space**:

$$\|u + v\| \leq \max\{\|u\|, \|v\|\}.$$

$\text{BALLS}_c^{(n)}$

- $\mathbb{K}((t))^n$ an **ultrametric space**:

$$\|u + v\| \leq \max\{\|u\|, \|v\|\}.$$

- A basis of clopen sets for the topology is given by **closed balls**, and their coordinate projections.

- A **closed ball** in $\mathbb{K}((t))^n$ is, for some $z \in \mathbb{K}((t))^n$, and $\gamma \in \Gamma^{>0}$,

$$B(z, \leq \gamma) = \{u \in \mathbb{K}((t))^n : \|u - z\| \leq \gamma\}.$$

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- The set $\mathrm{BALLS}_c^{(n)}$ of all closed balls is a definable family of sets, parametrised, say, by imaginary sort $(\mathbb{K}((t))^n \times \Gamma^{>0})/\sim$.

RV and $\mathrm{RV}^{(n)}$

- RV is the sort $\mathbb{K}((t))^\times/(1+\mathscr{M})$.
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Definition (The *n*-dimensional leading term structure)

is defined as $\mathrm{RV}^{(n)} := \mathbb{K}((t))^n/\sim$ where

$$x \sim y \Leftrightarrow (\|y - x\| < \|x\| \vee x = y).$$

The canonical map $\mathbb{K}((t))^n \to \mathrm{RV}^n$ is denoted by $\mathrm{rv}^{(n)}$.

- Suppose $B \subseteq \mathbb{K}((t))^n$ is a closed ball. Write **nnd** to emphasise "not necessarily definable".

Definition

An **nnd-isometry** on B is a bijection $f : B \to B$ such that for all $x_1, x_2 \in B$,

$$\|\phi(x_1) - \phi(x_2)\| = \|x_1 - x_2\|.$$

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Definition

An **nnd-risometry** on B is a bijection $\phi : B \to B$ such that for all $x_1, x_2 \in B$,

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For x_1, x_2 **distinct**, same as the condition

$$\|\phi(x_1) - \phi(x_2) - (x_1 - x_2)\| < \|x_1 - x_2\|.$$

Isometries vs Risometries vs Translations

- Suppose $B \subseteq \mathbb{K}((t))^n$ is a closed ball.

Definition

A **risometry** $\phi : B \to B$ is a bijection such that
for all $x_1, x_2 \in B$,

$$\mathrm{rv}^{(n)}(\phi(x_1) - \phi(x_2)) = \mathrm{rv}^{(n)}(x_1 - x_2).$$

For x_1, x_2 **distinct**, same as the condition

$$\|\phi(x_1) - \phi(x_2) - (x_1 - x_2)\| < \|x_1 - x_2\|.$$

- **Isometries** preserve the valuation of differences;
- **Risometries** preserve the "leading term" of differences;
- **Translations** precisely preserve differences.

- X is **nnd-risometric** to Y on $B \in \mathrm{BALLS}_c$ if there exists a risometry on B taking X to Y.

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Theorem (BW,Halupczok)

Let A be some parameters (from any sort) and suppose that $X, Y \subseteq \mathbb{K}((t))^n$ are $L(A)$ definable. Suppose $Q \subseteq \mathbb{K}((t))^m \times \mathrm{RV}^{(\ell)}$ is an $L(A)$-definable set parametrising closed balls $\{B_q\}_{q \in Q}$ in $\mathbb{K}((t))^n$. Then there is an $L(A)$-formula $\psi(z)$ such that, for each $q \in Q$,

$$\models \psi(q) \Leftrightarrow X \textit{ is nnd-risometric to } Y \textit{ on } B_q.$$

Prove inductively on ambient dimension n in tandem with

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$$\models \tau_r(q) \Leftrightarrow$$

$$\text{on } B_q, X \text{ is nnd-risometric with some subspace } \hat{V} \subseteq \mathbb{K}((t))^n$$

$$\text{such that } \mathrm{res}(\hat{V}) \subseteq \mathbb{K}^n \text{ is an } r\text{-dim } \mathbb{K}\text{-vector space.}$$

- Upshot: Given an $L(A)$-definable set X, the partition of $\mathrm{BALLS}_c^{(n)}$ into the balls on which X is "$=r$-translatable" is $L(A)$-definable (in the sense stated above).

Thank you for your attention!