Extending partial automorphisms of graphs

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Thank you Peter, time to party!

On the day of my PhD viva (2015)

EPPA: Extension Property for Partial Automorphisms

Definition (EPPA: Extension Property for Partial Automorphisms)

Let $\mathscr{C}$ be a class of finite structures. Here $G \leq H$ (induced) in $\mathscr{C}$.

- When **every** partial automorphism of G extends to an automorphism of H, H is called an **EPPA witness** for G.
- If every G in $\mathscr{C}$ has an EPPA witness in $\mathscr{C}$, say $\mathscr{C}$ **has EPPA**.

Theorem (E. Hrushovski (1992))

The class of finite graphs has EPPA.

Hence sometimes called the *Hrushovski Property*.

Examples: Subgraphs of finite homogeneous graphs

Definition (Homogeneous finite graph)

H **homogeneous**:
every partial automorphism of H extends to an automorphism;
$=$ H is an EPPA witness of itself.

Theorem (T. Gardiner (1976))

The finite homogeneous graphs are:

- *disjoint unions of cliques K_n, complements of these;*
- *The 5-cycle C_5;*
- *The rooks graph $L(K_{3,3})$ = line graph of complete bipartite graph $K_{3,3}$.*

Note: C_6, P_4, $K_1 \cup K_{1,2} \leq L(K_{3,3})$. So $L(K_{3,3})$ is an EPPA witness for them.

A motivation and generics

Theorem (E. Hrushovski (1992))

The class of finite graphs has EPPA.

A (final) combinatorial ingredient required in the proof of:

Theorem (W. Hodges, I. Hodkinson, D. Lascar, and S. Shelah (1993))

$\mathrm{Aut}\,\Gamma$ *of the countable random graph* Γ *has the* **small index property**.

- $G = \mathrm{Aut}\,\Gamma$ has a natural topology from S_ω: cosets of $G_{\bar{a}}$ are basic opens.
- The **small index property**: Every $H \leq G$ with $|G : H| < 2^\omega$ is open.
- Topology of G determined by its **abstract** group structure.

Definition

- Whenever $H \geq G$ are finite **graphs** such that *every* partial automorphism of G is the restriction of an automorphism of H, H is called an **EPPA witness** for G.
- The **EPPA numbers**:

$$\mathrm{eppa}(G) = \min\{|H| : H \text{ is an EPPA-witness for } G\},$$

$$\mathrm{eppa}(n) = \max\{\mathrm{eppa}(G) : |G| = n\}.$$

Theorem (E. Hrushovski (1992))

$$2^{n/2} \leq \mathrm{eppa}(n) \leq (2n2^n)! < \infty$$

Challenge (E. Hrushovski (1992))

Improve the bounds!

Upper bounds

Theorem (Herwig, Lascar (2000))

For every G with n vertices, m edges and maximum degree Δ we have that $\mathrm{eppa}(G) \leq \binom{\Delta n - m}{\Delta} \in n^{\mathcal{O}(\Delta)}$.
In particular, bounded degree graphs have polynomial EPPA numbers.
Witnesses are Δ**-set intersection graphs***:*
Johnson graphs or complements of Kneser graphs.

Corollary (Herwig, Lascar (2000))

$$\mathrm{eppa}(n) \leq \left(\frac{3en}{4}\right)^n.$$

Theorem (Evans,Hubička, Konečný, Nešetřil (2021))

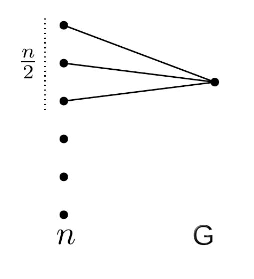
Valuation construction: $\mathrm{eppa}(n) \leq n2^{n-1}$.

A lower bound

Observation (B-W, Cameron,Hubička, Konečný (2025))

$$\Omega(2^n/\sqrt{n}) \leq \mathrm{eppa}(n).$$

Proof (basically Hrushovski'92 with a different graph).



- Every permutation of the left part is a partial automorphism of G.
- **Claim:** In every EPPA-witness, for every $S \in \binom{[n]}{n/2}$, there is a vertex connected to S and not to $[n] \setminus S$.
- Pick arbitrary $S \in \binom{[n]}{n/2}$.
- $\mathrm{eppa}(G) \geq \binom{n}{n/2} \in \Omega(2^n/\sqrt{n})$.

□

Observation

If G contains an independent set I and a vertex connected to exactly k members of I then $\mathrm{eppa}(G) \geq \binom{|I|}{k}$.

Corollary (B-W, Cameron,Hubička, Konečný (2025))

If G is triangle-free with maximum degree Δ then

$$\mathrm{eppa}(G) \in \Omega(n^{\Delta}).$$

Corollary (B-W, Cameron,Hubička, Konečný (2025))

Cycles have quadratic EPPA numbers, in fact:

$$\frac{1}{8}n^2 + o(n^2) \leq \mathrm{eppa}(C_n) \leq \frac{1}{2}n^2 + o(n^2).$$

What is the correct coefficient?

Summary

Theorem (B-W, Cameron,Hubička, Konečný (2025))

$$\Omega(2^n/\sqrt{n}) \leq \text{eppa}(n) \leq n2^{n-1},$$

the upper bound from the valuation graph construction of EHKN.

While some families have much smaller upper bounds:

1. (Induced) subgraphs of finite homogeneous graphs;
2. Cycles C_n have *eppa*(C_n) asymptotically quadratic (the upper bound coming from Johnson graphs).

Problem

When are these upper bounds attained?
When are the associated EPPA witnesses **smallest** *possible?*

Subgraphs of finite homogeneous graphs

Theorem (T. Gardiner (1976))

The finite homogeneous graphs are:

- *disjoint unions of cliques K_n, complements of these;*
- *The 5-cycle C_5;*
- *$L(K_{3,3})$, the line graph of complete bipartite graph $K_{3,3}$.*

Note: C_6, P_4, $K_1 \cup K_{1,2} \leq L(K_{3,3})$.

Exercise: Is $L(K_{3,3})$ a **smallest** EPPA witness for these graphs?

Tools: The automorphism groups

Observation

Suppose that H is an EPPA witness for G.
Then $\mathrm{Aut}(H)$ has a section isomorphic to $\mathrm{Aut}(G)$;
in particular, $|\mathrm{Aut}(G)|$ divides $|\mathrm{Aut}(H)|$.

Proof.

From the definition of EPPA witness, we see that the setwise stabiliser of $V(G)$ in $\mathrm{Aut}(H)$ induces $\mathrm{Aut}(G)$ on $V(G)$. □

Tools: The automorphism groups

Lemma (B-W, Cameron,Hubička, Konečný)

Let G be a graph, and H an EPPA witness for G with the smallest number of vertices and, subject to that, the smallest number of edges. Suppose that neither G nor G' is a disjoint union of complete graphs.

1. *H is vertex-transitive.*
2. *H is arc-transitive (arc = oriented edge).*
3. *Either H is **vertex-primitive**, or the vertex set of G contains at most one point of any block of imprimitivity for $\mathrm{Aut}(H)$.*

So minimality of an EPPA witness H can sometimes (say when $|G| < |H| < 2|G|$) can be **verified** by considering possibilities of primitive groups of degree $d, |G| < d < |H|$.

Scarcity of primitive permutation groups

Degree	Nr Permutation Groups *OEIS* : *A000019*	Nr Primitive Groups *OEIS* : *A000638*
1	1	1
2	1	1
3	2	2
4	4	2
5	11	5
6	19	4
7	56	7
8	96	7
9	296	11
10	554	9
11	1593	8
12	3094	6
13	10723	9
14	20832	4

Sample argument

Proposition

$L(K_{3,3})$ is a smallest EPPA witness for C_6.

Proof.

- $|L(K_{3,3})| = 9$ and $|C_6| = 6$;
- by the lemma, a smaller EPPA witness has vertex-primitive automorphism group of degree 7 or 8 with $\mathrm{Aut}(C_6)$ (order 12) as a section.
- After checking the few possibilities (e.g. with GAP libraries for primitive groups), see that there is no such primitive group.

□

Very small EPPA witnesses

Proposition (B-W, Cameron, Hubička, Konečný (2025))

Let G be a graph on n vertices, which has a smallest EPPA-witness H on fewer than $(5/4)n$ vertices. Then H is homogeneous.

Proof.

k-homogeneous: any isomorphism between induced subgraphs on at most k vertices extends to an automorphism. We use two ingredients in the proof:

(a) (Π. Neumann's Separation Lemma). Let A and B be subsets of the domain of a transitive permutation group G of degree n. If $|A| \cdot |B| < n$, then there exists $g \in G$ such that $Ag \cap B = \emptyset$.

(b) (P. Cameron). A 5-homogeneous graph is homogeneous.

… □

Small EPPA witnesses

Theorem (B-W, Cameron, Hubička, Konečný (2025))

Let G be a graph on n vertices, and H a smallest EPPA-witness for G with fewer than $2n$ vertices. Then $\mathrm{Aut}(H)$ is a rank 3 permutation group on $V(H)$.

Proof.

Using P. Neumann's Separation Lemma with 2 replacing 5, get H is 2-homogeneous: $\mathrm{Aut}(H)$ is transitive on vertices, oriented edges, and oriented non-edges; the definition of rank 3. □

Work in progress with S. Brenner

Classifying the graphs G on n vertices which have an EPPA witness on at most $2n$ vertices.

Vertex primitive case: almost all cycles

Corollary (B-W, Cameron, Hubička, Konečný (2025))

The n-cycles C_n have $\mathrm{eppa}(C_n)$ *asymptotically quadratic.*

Theorem (B-W, Cameron, Hubička, Konečný)

For all but finitely many n, a smallest EPPA witness of C_n is the Johnson graph $J(n,2)$ on $\binom{n}{2}$ vertices.

This means almost always attaining the bound from the Herwig-Lascar construction with the graph on 2-sets defined by

$$u \sim v := |u \cap v| = 1.$$

Vertex primitive case: almost all cycles

Theorem (B-W, Cameron, Hubička, Konečný)

For all but finitely many n, a smallest EPPA witness of C_n is the Johnson graph $J(n,2)$ on $\binom{n}{2}$ vertices (including $n = 7$).

Via our Main Lemma a major application of the following is used.

Theorem (A. Maróti (2002) applying the greatness of CFSG)

Let G be a primitive group of degree N which is not S_N or A_N. Then one of the following possibilities occurs:

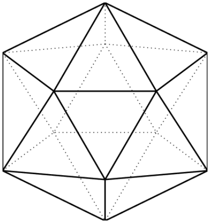
(a) *For some integers m, k, l, we have $N = \binom{m}{k}^l$, and G is a subgroup of $S_m \wr S_l$, where S_m is acting on k-subsets of $\{1, \ldots, m\}$, and G contains $(A_m)^l$;*

(b) *G is M_{11}, M_{12}, M_{23} or M_{24} in its natural 4-transitive action;*

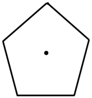
(c) $|G| \leq N \cdot \prod_{i=0}^{\lfloor \log_2 N \rfloor - 1} (N - 2^i)$.

Imprimitive case: Double covers

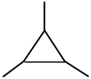
- What happens in the Aut H **imprimitive** case of the Main Lemma?
- G is embedded as a transversal across the blocks of Aut H.
- In the case of blocks of size 2: H is a double cover of G.
- This took us on a tour though two-graphs, Seidel switching, and connected metrically-homogeneous finite graphs (classified by Peter in 1980) which we don't have time for today.
- One of the latter is the **isocahedron** with group $A_5 \times C_2$:



Isocahedron graph (from D. Mugnolo, M. Plümer)



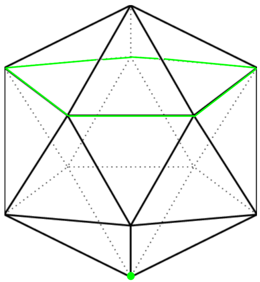
The **spokeless wheel** W.



The **legged triangle** L.

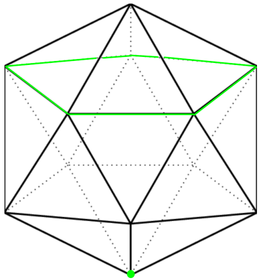
Some imprimitive EPPA double covers. (B-W, Cameron, Hubička, Konečný)

The icosahedron graph (on 12 vertices) is a **smallest** EPPA witness for both the spokeless wheel and the legged triangle (on 6 vertices).



Isocahedron graph with embedded W (edited from D. Mugnolo, M. Plümer)

Isocahedron here is significantly better than Herwig-Lascar:
$|HL(W)| = |J(6.2 - 5, 2)| = \binom{7}{2} = 21$
$|HL(L)| = \binom{12}{3} = 220.$



Isocahedron graph with embedded W (edited from D. Mugnolo, M. Plümer)

Isocahedron **smallest** EPPA witness for W.

- Note $|\operatorname{Aut} W| = |D_{10}| = 10$ and consider H with $|H| < 12$.
- By Main Lemma, $\operatorname{Aut}(H)$ primitive of degree n with $6 < n < 12$.
- Only possible $\operatorname{Aut}(H)$ with order divisible by 10 is S_5 acting on 10.
- Corresponding graphs are the Petersen graph or complement $L(K_5)$.
- Neither embeds W so they are not EPPA witnesses.

Big Open Question: Herwig–Lascar (2000)

Does the class of finite tournaments have EPPA?

Questions on EPPA numbers (See paper for more)

- We still have a factor $n^{3/2}$ between the lower bound and the upper bound for $\mathrm{eppa}(n)$. What is the correct bound?
- Let G be the lower-bound graph from earlier, we calculated $\Omega(2^n/\sqrt{n}) \leq \mathrm{eppa}(G)$. What is the EPPA number of G?
- What are the EPPA numbers of Payley graphs? Half-graphs?

We have got a lot out of considering how $\mathrm{Aut}\, G$ compares to $\mathrm{Aut}\, H$.

Speculation

$\mathrm{PIso}(G)$ and $\mathrm{PIso}(H)$ are **inverse monoids** and perhaps their relationship could say a lot more...

Thank you for your attention!

From images.ansharimages.com

D. B-W, P. J. Cameron, J. Hubička, M. Konečný, **EPPA numbers of graphs**, Journal of Combinatorial Theory Series B, Volume 170, Pages 203–224 (2025).
EPPA numbers of graphs II, coming soon to Arxiv.

D. B-W, S. Brenner, **EPPA witnesses with two blocks of imprimitivity on edges**, coming soon to Arxiv and EUROCOMB25.