EPPA numbers of graphs

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EPPA: Extension Property for Partial Automorphisms

Definition (EPPA: Extension Property for Partial Automorphisms)

Let $\mathscr{C}$ be a class of finite structures. Here $G \leq H$ (induced) in $\mathscr{C}$.

- When **every** partial automorphism of G extends to an automorphism of H, H is called an **EPPA witness** for G.
- If every G in $\mathscr{C}$ has an EPPA witness in $\mathscr{C}$, say $\mathscr{C}$ **has EPPA**.

Theorem (E. Hrushovski (1992))

The class of finite graphs has EPPA.

Hence sometimes called the *Hrushovski Property*.

Examples: Subgraphs of finite homogeneous graphs

Definition (Homogeneous graph)

H is **homogeneous** if every partial automorphism of H extends to an automorphism.

Theorem (T. Gardiner (1976))

The finite homogeneous graphs are:

- *disjoint unions of cliques K_n, complements of these;*
- *The 5-cycle C_5;*
- *The rooks graph $L(K_{3,3})$ = line graph of complete bipartite graph $K_{3,3}$.*

Note: $C_6, P_4, K_1 \cup K_{1,2} \leq L(K_{3,3})$. So $L(K_{3,3})$ is an EPPA witness for them.

A motivation and generics

Theorem (E. Hrushovski (1992))

The class of finite graphs has EPPA.

A (final) combinatorial ingredient required in the orginal proof of:

Theorem (W. Hodges, I. Hodkinson, D. Lascar, and S. Shelah (1993))

The automorphism group of the random graph Γ has the small index property.

One of the now standard methods to prove the existence of *ample generic automorphisms* of the Fraïssé limit of $\mathscr{C}$ involves proving that $\mathscr{C}$ has EPPA. Although the proof of J. K. Truss (1992) of generics in $\operatorname{Aut}\Gamma$ does not use (full) EPPA.

EPPA for some other structures

Theorem (Kechris, Rosendal (2007))

The class of all finite substructures of a countable homogeneous structure M has EPPA if and only if $\mathrm{Aut}(M)$ can be written as the closure of a chain of compact subgroups. (Implies $\mathrm{Aut}(M)$ amenable).

Theorem (Herwig, Lascar (2000))

Let L be a finite relational language and T a finite set of finite L-structures. Then the class of T-free L-structures has the EPPA.

Examples: Directed graphs, k-hypergraphs,...
Not covered: Tournaments,...
Non-examples: Ordered structures.

Definition

- Whenever $H \geq G$ are finite **graphs** such that *every* partial automorphism of G is the restriction of an automorphism of H, H is called an **EPPA witness** for G.
- The **EPPA numbers**:

$$\mathrm{eppa}(G) = \min\{|H| : H \text{ is an EPPA-witness for } G\},$$

$$\mathrm{eppa}(n) = \max\{\mathrm{eppa}(G) : |G| = n\}.$$

Theorem (E. Hrushovski (1992))

$$2^{n/2} \leq \mathrm{eppa}(n) \leq (2n2^n)! < \infty$$

Challenge (E. Hrushovski (1992))

Improve the bounds!

Upper bounds

Theorem (Herwig, Lascar (2000))

For every G with n vertices, m edges and maximum degree Δ we have that $\mathrm{eppa}(G) \leq \binom{\Delta n - m}{\Delta} \in n^{\mathcal{O}(n)}$.
In particular, bounded degree graphs have polynomial EPPA numbers.
Witnesses are Δ**-set intersection graphs**.

Corollary (Herwig, Lascar (2000))

$$\mathrm{eppa}(n) \leq \left(\frac{3en}{4}\right)^n.$$

Theorem (Evans,Hubička, Konečný, Nešetřil (2021))

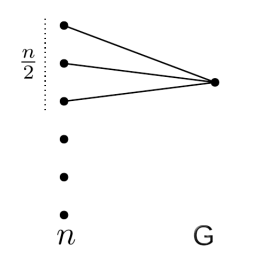
Valuation construction: $\mathrm{eppa}(n) \leq n2^{n-1}$.

A lower bound

Observation (B-W, Cameron,Hubička, Konečný (2025))

$$\Omega(2^n/\sqrt{n}) \leq \mathrm{eppa}(n).$$

Proof (basically Hrushovski'92 with a different graph).



- Every permutation of the left part is a partial automorphism of G.
- **Claim:** In every EPPA-witness, for every $S \in \binom{[n]}{n/2}$, there is a vertex connected to S and not to $[n] \setminus S$.
- Pick arbitrary $S \in \binom{[n]}{n/2}$.
- $\mathrm{eppa}(G) \geq \binom{n}{n/2} \in \Omega(2^n/\sqrt{n})$.

□

Observation

If G contains an independent set I and a vertex connected to exactly k members of I then $\mathrm{eppa}(G) \geq \binom{|I|}{k}$.

Corollary (B-W, Cameron,Hubička, Konečný (2025))

If G is triangle-free with maximum degree Δ then

$$\mathrm{eppa}(G) \in \Omega(n^{\Delta}).$$

Corollary (B-W, Cameron,Hubička, Konečný (2025))

Cycles have quadratic EPPA numbers, in fact:

$$\frac{1}{8}n^2 + o(n^2) \leq \mathrm{eppa}(C_n) \leq \frac{1}{2}n^2 + o(n^2).$$

What is the correct coefficient?

Summary

Theorem (B-W, Cameron,Hubička, Konečný (2025))

$$\Omega(2^n/\sqrt{n}) \leq \mathrm{eppa}(n) \leq n2^{n-1},$$

the upper bound from the valuation graph construction of EHKN.

While some families have much smaller upper bounds:

1. (Induced) subgraphs of finite homogeneous graphs;
2. Cycles C_n have $eppa(C_n)$ asymptotically quadratic (the upper bound coming from Kneser graphs).

Problem

When are these upper bounds attained?
When are the associated EPPA witnesses smallest possible?

Subgraphs of finite homogeneous graphs

Theorem (T. Gardiner (1976))

The finite homogeneous graphs are:

- *disjoint unions of cliques K_n, complements of these;*
- *The 5-cycle C_5;*
- *$L(K_{3,3})$, the line graph of complete bipartite graph $K_{3,3}$.*

Note: C_6, P_4, $K_1 \cup K_{1,2} \leq L(K_{3,3})$.

Exercise: Is $L(K_{3,3})$ a **smallest** EPPA witness for these graphs?

Tools: The automorphism groups

Observation (B-W, Cameron,Hubička, Konečný)

Suppose that H is an EPPA witness for G. Then $\mathrm{Aut}(H)$ has a section (a quotient of a subgroup) isomorphic to $\mathrm{Aut}(G)$;
in particular, $|\mathrm{Aut}(G)|$ divides $|\mathrm{Aut}(H)|$.

Proof.

From the definition of EPPA witness, we see that the setwise stabiliser of $V(G)$ in $\mathrm{Aut}(H)$ induces $\mathrm{Aut}(G)$ on it. □

Tools: The automorphism groups

Lemma (B-W, Cameron, Hubička, Konečný)

Let G be a graph, and H an EPPA witness for G with the smallest number of vertices and, subject to that, the smallest number of edges. Suppose that neither G nor G' is a disjoint union of complete graphs.

1. *H is vertex-transitive.*
2. *H is arc-transitive (arc = oriented edge).*
3. *Either H is vertex-primitive, or the vertex set of G contains at most one point of any block of imprimitivity for $\mathrm{Aut}(H)$. In the latter case, the number of vertices of the EPPA witness is at least twice the number of vertices of G.*

So minimality of an EPPA witness H can sometimes (say when $|G| < |H| < 2|G|$) can be verified by considering possibilities of primitive groups of degree $d, |G| < d < |H|$.

Scarcity of primitive permutation groups

Degree	Nr Permutation Groups *OEIS* : *A000019*	Nr Primitive Groups *OEIS* : *A000638*
1	1	1
2	1	1
3	2	2
4	4	2
5	11	5
6	19	4
7	56	7
8	96	7
9	296	11
10	554	9
11	1593	8
12	3094	6
13	10723	9
14	20832	4

Sample argument

Proposition

$L(K_{3,3})$ is a smallest EPPA witness for C_6.

Proof.

- $|L(K_{3,3})| = 9$ and $|C_6| = 6$;
- by the lemma, a smaller EPPA witness has vertex-primitive automorphism group of degree 7 or 8 with $\mathrm{Aut}(C_6)$ (order 12) as a section.
- After checking the few possibilities (e.g. with GAP libraries), see that there is no such primitive group.

□

Very small EPPA witnesses

Proposition (B-W, Cameron, Hubička, Konečný (2025))

Let G be a graph on n vertices, which has a smallest EPPA-witness H on fewer than $(5/4)n$ vertices. Then H is homogeneous.

Proof.

k-homogeneous: any isomorphism between induced subgraphs on at most k vertices extends to an automorphism. We use two ingredients in the proof:

(a) (P. Neumann's Separation Lemma). Let A and B be subsets of the domain of a transitive permutation group G of degree n. If $|A| \cdot |B| < n$, then there exists $g \in G$ such that $Ag \cap B = \emptyset$.

(b) (P. Cameron). A 5-homogeneous graph is homogeneous.

...

□

Small EPPA witnesses

Theorem (B-W, Cameron,Hubička, Konečný)

Let G be a graph on n vertices, and H a smallest EPPA-witness for G with fewer than $2n$ vertices. Then $\mathrm{Aut}(H)$ is a rank 3 permutation group on $V(H)$.

Proof.

Using P. Neumann's Separation Lemma with 2 replacing 5, get H is 2-homgeneous: $\mathrm{Aut}(H)$ is transitive on vertices, oriented edges, and oriented non-edges; the definition of rank 3. □

Work in progress with S. Brenner

Classifying the graphs G on n vertices which have an EPPA witness on fewer than $2n$ vertices. (When complete will generalize Gardiner's classification of finite homogeneous graphs.)

Two-graphs and double covers

EPPA double covers

Suppose $|G| = n$ and an EPPA witness has $|H| = 2n$ with $\mathrm{Aut}(H)$ imprimitive.

Turns out very special graphs G have EPPA-covers H arising from two-graphs corresponding to the "switching class" of G.

Definition

A **two-graph** is a pair (X, T), where X is a set and T a collection of 3-hyperedges on X such that any 4-subset of X contains an even number of hyperedges from T.

Example: $\mathscr{T}(G)$

Given graph G, the 3-hypergraph $\mathscr{T}(G) = (G, T)$ where $xyz \in T$ if xyz induce an **odd** number of edges of G is a **two-graph**.

Definition

Graphs G_1 and G_2 on the same vertex set X are **switching-equivalent** if there is a subset Y of X such that G_1 and G_2 have the same edges within our outside Y and complementary edges between Y and $X \setminus Y$.

Fact: G_1 and G_2 are switching equivalent if $\mathcal{T}(G_1) \cong \mathcal{T}(G_2)$.

Definition

The **double cover** $D(G)$ of the graph G is a graph on a set $\hat{G}$ with a two-to-one surjection τ to G such that

- points with the same image under τ are not adjacent in $D(G)$;
- if $\tau(x_1) = \tau(x_2) \neq \tau(y_1) = \tau(y_2)$, there are two disjoint edges between $\{x_1, x_2\}$ and $\{y_1, y_2\}$.
- $xy \in E(G) \leftrightarrow x_1y_1$ and x_2y_2 in $E(\hat{G})$.

Flipping over $y \in G$ is the same as switching w.r.t. $\{y\}$.

Theorem (B-W, Cameron, Hubička, Konečný)

Let (X, T) be the two-graph corresponding to a switching class S. Let D be the corresponding double cover. Then the following are equivalent:

(a) *(X, T) is a homogeneous two-graph;*
(b) *D is homogeneous (as a structure with a partition into parts of size 2 in addition to the graph structure);*
(c) *D is an EPPA-witness for partial isomorphisms between graphs in S. In particular D* **is an EPPA-witness for any graph in** *S.*

Natural question: which are the homogeneous two-graphs?

Theorem (B-W, Cameron, Hubička, Konečný)

Apart from the complete and null two-graphs, there are just two homogeneous two-graphs, on 6 and 10 points respectively.

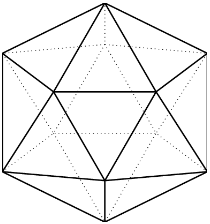
All EPPA double covers from homogeneous two-graph on 6 points.

The double cover is the **icosahedron graph**.

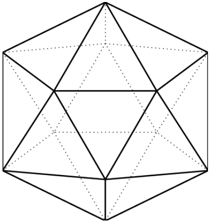
Four graphs in the switching class, two complementary pairs:

- a 5-cycle with an isolated vertex (the **spokeless wheel**), and its complement; automorphism group dihedral of order 10;
- a triangle with a pendant edge at each vertex (the **legged triangle**) and its complement, automorphism group dihedral of order 6.

Here the icosahedron (12 vertices) is also a **smallest** EPPA witness.



Isocahedron graph (from D. Mugnolo, M. Plümer)



Isocahedron graph (from D. Mugnolo, M. Plümer)

Isocahedron **smallest** EPPA witness for W.

- E.g. for the spokeless wheel W: consider H with $|H| < 12$.
- By Lemma, $\mathrm{Aut}(H)$ primitive of degree n with $6 < n < 12$.
- Only possible $\mathrm{Aut}(H)$ with order divisible by 10 is S_5 acting on 10.
- Corresponding graphs are the Petersen graph or complement $L(K_5)$.
- Neither embeds W so they are not EPPA witnesses.

Almost all cycles

Theorem (B-W, Cameron, Hubička, Konečný)

For all but finitely many n, the smallest EPPA witness of the n-cycle graph has $\binom{n}{2}$ vertices (attaining bound from Herwig-Lascar construction).

Theorem (A. Maróti (2002))

Let G be a primitive group of degree N which is not S_N or A_N. Then one of the following possibilities occurs:

(a) *For some integers m, k, l, we have $N = \binom{m}{k}^l$, and G is a subgroup of $S_m \wr S_l$, where S_m is acting on k-subsets of $\{1, \ldots, m\}$, and G contains $(A_m)^l$;*

(b) *G is M_{11}, M_{12}, M_{23} or M_{24} in its natural 4-transitive action;*

(c) $|G| \leq N \cdot \prod_{i=0}^{\lfloor \log_2 N \rfloor - 1} (N - 2^i)$.

Big Open Question: Herwig–Lascar (2000)

Does the class of finite tournaments have EPPA?

Questions on EPPA numbers (See paper for more)

- We still have a factor $n^{3/2}$ between the lower bound and the upper bound for $\mathrm{eppa}(n)$. What is the correct bound?
- Let G be the lower-bound graph from earlier, we calculated $\Omega(2^n/\sqrt{n}) \leq \mathrm{eppa}(G)$. What is the EPPA number of G?
- Can you prove similar lower bounds on EPPA numbers for planar graphs?
- What are the EPPA numbers of Payley graphs? Half-graphs?

Using a variation of the valuation construction, we calcuated that K_k-free graphs have K_k-free EPPA witnesses on at most $2^{2^{\mathcal{O}(kn)}}$ vertices.

- What is a better lower bound for K_k-free EPPA witnesses of K_k-free (non-subhomogeneous) graphs?

Thank you for your attention!

From images.ansharimages.com

- D. B-W, P. J. Cameron, J. Hubička, M. Konečný, **EPPA numbers of graphs**, Journal of Combinatorial Theory Ser. B, Volume 170, Pages 203–224 (2025).
- **EPPA numbers of graphs II**, coming soon to Arxiv.